LITERATURE CITED

1. V. P. Silin, "Light absorption by a turbulent plasma," Usp. Fiz. Nauk, 145, No. 2 (1985).
2. Yu. P. Raizer, The Laser Spark and Discharge Propagation [in Russian], Nauka, Moscow (1974).
3. B. B. Kadomtsev, Collective Phenomena in Plasma [in Russian], Nauka, Moscow (1988).
4. L. A. Artsimovich and R. Z. Sagdeev, Plasma Physics for Physicists [in Russian], Nauka, Moscow (1979).
5. G. M. Zaslavskii and R. Z. Sagdeev, Introduction to Nonlinear Physics [in Russian], Nauka, Moscow (1988).
6. G. M. Zaslavskii, Stochasticity in Dynamic Systems [in Russian], Nauka, Moscow (1984).
7. V. A. Danilychev and V. D. Zvorykin, "Interaction of CO_2-laser radiation with a target in gases," Tr. FIAN, 142, (1983).

ASYMPTOTE OF THE NAVIER–STOKES EQUATION SOLUTION IN THE VICINITY OF A BOUNDARY ANGLE

V. A. Kondrat'ev

UDC 532.516

In a study of single-sided limitations for the Navier–Stokes equations, [1] considered the function $\psi(r, \varphi)$, which satisfies the equation

$$\Delta\Delta\psi = 0,\ r < \varepsilon,\ -\pi < \varphi < 0 \tag{1}$$

(where $\varepsilon > 0$ is a constant) with boundary conditions

$$\psi = 0,\ \Delta\psi = 0,\ \varphi = 0,\ 0 < r < \varepsilon,$$
$$\psi = 0,\ \frac{\partial\psi}{\partial\varphi} = r,\ \ \varphi = -\pi,\ \ 0 < r < \varepsilon.$$

Here (r, φ) is a planar polar coordinate system and Δ is the Laplace operator. In addition we assume the function belongs to the Sobolev space W_2^2 in the semicircle $S_\varepsilon = \{(r, \varphi) : r < \varepsilon, -\pi < \varphi < 0\}$. Using the method developed in [2, 3] the authors presented the expression

$$\psi = -r\sin\varphi + Ar^{3/2}\left(\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2}\right) + O(r^2\ln r) \tag{2}$$

for $r \to 0$, $-\pi < \varphi < 0$, $A =$ const, which is dependent on ψ. Asymptotic representations of $\partial\psi/\partial r$, $\partial\psi/\partial\varphi$, $\Delta\psi$ can be obtained from Eq. (2) by formal differentiation. In fact, Eq. (2) can be refined: for ψ one can expand in an asymptotic series [2, 4]

$$\psi = -r\sin\varphi t\sum_{j=3}^{\infty} A_j r^{j/2}\Phi_j(\varphi),\quad A_j = \text{const}, \tag{3}$$

where Φ_j are eigenfunctions, normalized in $L_2[-\pi, 0]$, of the problem

$$\frac{1}{4}j^2\left(\frac{j}{2}-2\right)^2\Phi + \frac{j^2}{2}\Phi''' + \Phi^{IV} = 0,$$
$$-\pi < \varphi < 0,\ \Phi(-\pi) = \Phi(0) = 0,\ \Phi'(-\pi) = \Phi''(0) = 0. \tag{4}$$

Equation (3) is asymptotic in the sense that no matter what the value of N, the estimates

$$\left|D^\alpha(\psi) + r\sin\varphi - \sum_{j=3}^{N} A_j r^{j/2}\Phi_j(\varphi)\right| = O(r^{(N+1)/2-|\alpha|})$$

are valid as $r \to 0$ for all α. Here $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}$; $|\alpha| = \alpha_1 + \alpha_2$. Note that Eq. (4) with constant coefficients is easily solved and the eigenfunctions of Eq. (4) can be written explicity; Equation (3) is a special case of a more general expression which gives the asymptotic representation of a boundary problem for an arbitrary elliptic equation in the vicinity of an angular point on the region's boundary. It follows from Eq. (3) that in Eq. (2) the

Moscow. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 6, pp. 38-40, November-December, 1992. Original article submitted November 22, 1991.

 0021-8944/92/3306-0802$12.50

residual term can be replaced by $O(r^2)$, and its second derivatives are finite. In [4], which was used in [1] in deriving Eq. (2), it was not Eq. (1), but the nonlinear Navier–Stokes equation

$$\nu\Delta\Delta u + \frac{\partial u}{\partial x}\frac{\partial \Delta u}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial \Delta u}{\partial x} = 0, \quad r < \varepsilon, \quad 0 < \varphi < 2\pi \tag{5}$$

which was studied with boundary conditions

$$u = \frac{\partial u}{\partial \varphi} = 0, \quad \varphi = 0, \quad \varphi = 2\pi, \quad 0 < r < \varepsilon. \tag{6}$$

It was assumed that $u \in W_2^2(S_\varepsilon)$, $S_\varepsilon = \{x : 0 < \varphi < 2\pi, \; 0 < r < \varepsilon\}$. We will call the generalized solution of Eqs. (5), (6) the function $u(x) \in W_2^2(S_\varepsilon)$, satisfying boundary conditions (6), and such that

$$\nu \int\limits_{S_\varepsilon} \left[\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 \psi}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 \psi}{\partial x_2^2}\right] dx_1 dx_2 + \int\limits_{S_\varepsilon} \Delta u \left(\frac{\partial \psi}{\partial x_1}\frac{\partial u}{\partial x_2}\left(-\frac{\partial \psi}{\partial x_2}\frac{\partial u}{\partial x_1}\right) dx_1 dx_2 = 0$$

for any $\psi(x) \in \overset{\circ}{W}{}_2^2(S_\varepsilon)$.

In reality, the representation of Eq. (2) with residual term of the order of $O(r^2)$ can also be obtained for the solution of Eqs. (5), (6). In doing this we will make use of results from estimates $L_p(1 < p < \infty)$ of the boundary problem solutions of [5], which were not known at the time that [3] appeared. It was proved in [5] that if $u(x)$ is a generalized solution of the equation

$$\Delta\Delta u = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

in S_ε, satisfying boundary conditions (6),

$$u(x) \in W_q^2(S_\varepsilon), \quad q \geqslant 2, \quad \int\limits_{S_\varepsilon} |f_i|^p dx_1 dx_2 < \infty, \quad p > 1, \quad i = 1, 2,$$

then

$$u(x) = Ar^{3/2}\left(\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2}\right) + u_0(x), \tag{7}$$

where

$$\|u_0\|^2_{W_p^3(S_{\varepsilon/2})} \leqslant C\left[\sum_{i=1}^{2} \|f_i\|^2_{L_p(S_\varepsilon)} + \|u\|^2_{L_2(S_\varepsilon)}\right], \quad 1 < p < 2,$$

and

$$u(x) = Ar^{3/2}\left(\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2}\right) + Br^2 \sin^2\varphi + u_0(x). \tag{8}$$

Here

$$\|u_0(x)\|_{W_p^3(S_{\varepsilon/2})} \leqslant C\left[\|u\|_{L_2(S_\varepsilon)} + \sum_{i=1}^{2} \|f_i\|_{L_p(S_\varepsilon)}\right].$$

Moreover,

$$|A| \leqslant C\left[\|u\|_{L_2(S_\varepsilon)} + \sum_{i=1}^{2} \|f_i\|_{L_p(S_\varepsilon)}\right] \text{ for } 1 < p < 2,$$

$$|A| + |B| \leqslant C\left[\|u\|_{L_2(S_\varepsilon)} + \sum_{i=1}^{2} \|f_i\|_{L_p(S_\varepsilon)}\right] \text{ for } 2 < p < 3.$$

This result can be used to study the problem of Eqs. (5) and (6).

Theorem. If $u(x) \in W_2^2(S_\varepsilon)$ is a solution of the problem of Eqs. (5), (6), then u(x) has the form of Eq. (8), where

$$\sum_{0 \leqslant k_1+k_2=k \leqslant 2} \left| \frac{\partial^k u_0(x)}{\partial x_1^{k_1} \partial x_2^{k_2}} \right| \leqslant C \|u\|_{W_2^2(S_\varepsilon)}, \tag{9}$$

$x \in S_{\varepsilon/4}$, C = const, of which the solution is independent.

Proof. Equation (5) can be written in the form

$$\nu \Delta\Delta u = -\frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_1} \Delta u\right) + \frac{\partial}{\partial x_1}\left(\Delta u \frac{\partial u}{\partial x_2}\right) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}. \tag{10}$$

Here $f_1 \in L_s(\Omega_\varepsilon)$, $f_2 \in L_s(\Omega_\varepsilon)$ for any s < 2, since $\Delta u \in L_2(\Omega_\varepsilon)$, grad $u \in L_p(\Omega_\varepsilon)$ for any p < ∞. Thus, Eq. (7) is valid for u(x), where $u_0(x)$ satisfies the inequality

$$\sum_{k_1+k_2=3} \left\| \frac{\partial^3 u_0}{\partial x_1^{k_1} \partial x_2^{k_2}} \right\|_{L_p(S_{\varepsilon/2})} \leqslant C_p \|u\|_{W_2^2(S_\varepsilon)} \quad \forall p < 2. \tag{11}$$

From Eq. (11) and the Sobolev inclusion theorem it follows that $u_0 \in W_q^2(S_{\varepsilon/2})$, $u_0 \in C^1(S_{\varepsilon/2})$ for any q. Representing u in the form of Eq. (7) in Eq. (10), for u_0 we obtain

$$\nu \Delta\Delta u_0 = \frac{\partial}{\partial x_1} F_1 + \frac{\partial}{\partial x_2} F_2,$$

where $F_1 \in L_p(S_{\varepsilon/2})$, $F_2 \in L_p(S_{\varepsilon/2})$ for any p < 4 and

$$\|F_1\|_{L_p(S_{\varepsilon/2})} + \|F_2\|_{L_p(S_{\varepsilon/2})} \leqslant C \|u\|_{W_2^2(S_\varepsilon)}.$$

Using Eq. (8) for u_0, we have

$$u_0 = A_1 r^{3/2}\left(\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2}\right) + B_1 r^2 \sin^2\varphi + v_0(x). \tag{12}$$

Here $v_0(x) \in W_p^3(S_{\varepsilon/4})$ for any p < 4. From the Sobolev inclusion theorem we now know that $v_0 \in C^2(S_{\varepsilon/4})$. Substituting u_0 in the form of Eq. (12) in Eq. (7) we find the required Eq. (9), where $v_0 \in C^2(S_{\varepsilon/4})$.

LITERATURE CITED

1. K. Baiokki and V. V. Pukhnachev, "Problems with single-sided limitations for the Navier–Stokes equations and the dynamic boundary angle problem", Prikl. Mekh. Tekh. Fiz., No. 2 (1990).
2. V. A. Kondrat'ev, "Boundary problems for elliptical equations in regions with conical or angular points," Tr. Mosk. Matem. Obshch., 16, (1967).
3. V. A. Kondrat'ev, "Asymptote of Navier–Stokes equation solutions in the vicinity of an angular boundary point", Prikl. Matem. Mekh., 31, No. 1 (1967).
4. V. A. Kondrat'ev and O. A. Oleinik, "Boundary problems for equations with partial derivatives in non-smooth regions," Usp. Matem. Nauk, 38, No. 2 (1983).
5. V. G. Maz'ya and B. A. Plamenevskii, "L_p estimates, Hölder classes, and the Miranda–Agmona principle for solutions of elliptical boundary problems in regions with singular points on the boundary," Math. Nachr., 81, 25 (1978).